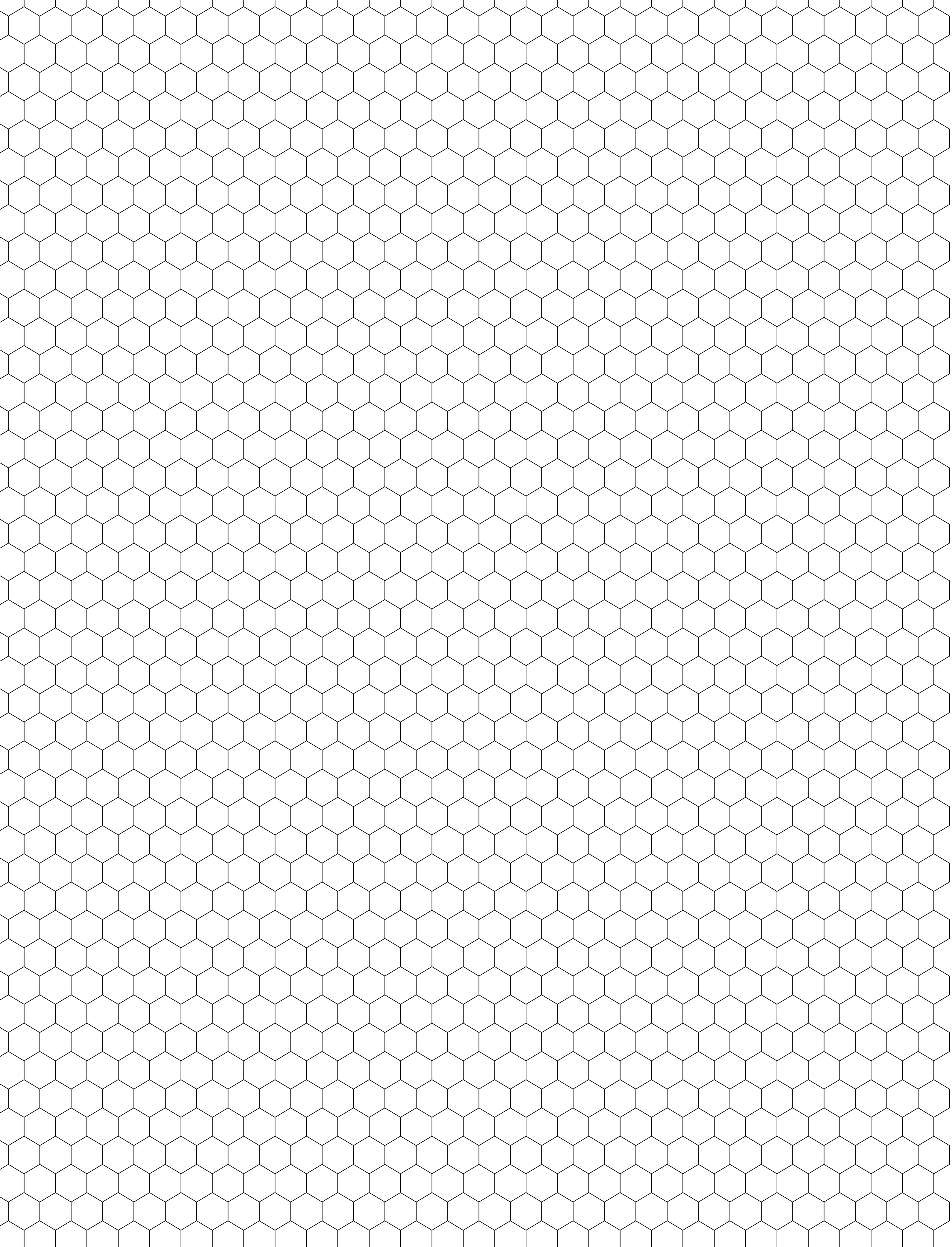
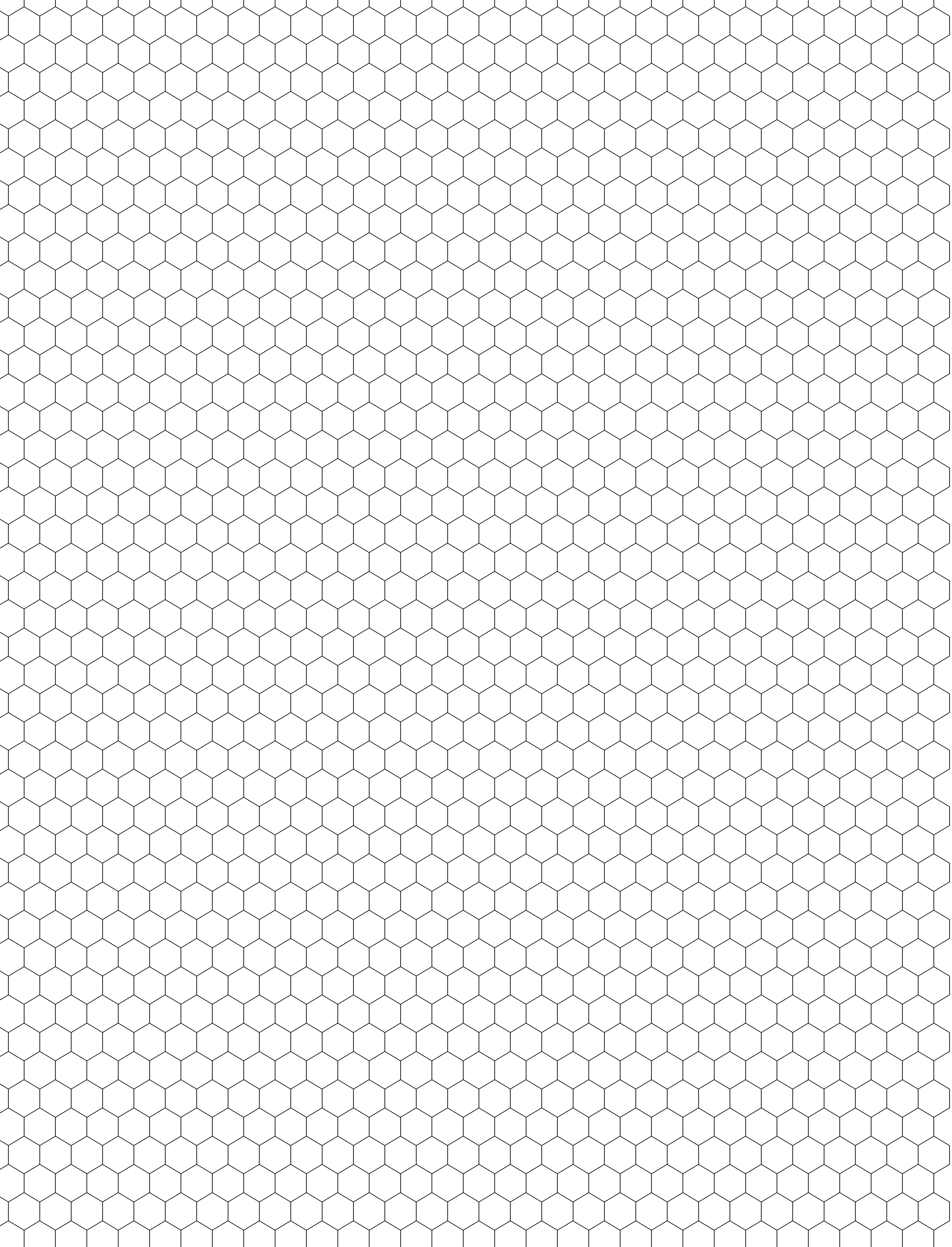




thanks Bjarne Toft, Jakub Pawlewicz, Kenny Young, Brad Thiessen, Broderick Arneson, Philip Henderson, Aja Huang, Martin Müller, NSERC 2/3 2981241/5200300 57

Rex, or **Reverse Hex**: whoever connects their two sides loses. For the $n \times n$ board, Winder gave a strategy-stealing proof that the first (second) player can win if n is even (odd). Lagarias and Sleator showed that the loser can prolong the game so that the board is filled.

Cylindrical Hex, or **CylHex**: $c \times h$ board is wrapped around a cylinder with circumference c, height h. **Updown** wins by connecting top to bottom. **Around** wins by encircling cylinder.

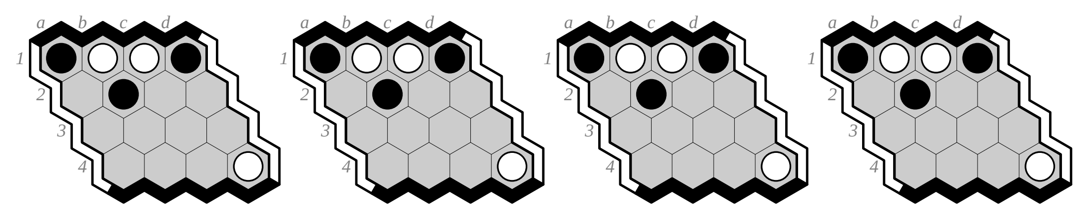
Random Hex: each player, on each turn, makes uniform random move, among all possible moves. Let $p_1(n)$ be the probability that the first player wins random $n \times n$ Hex.

Quiz

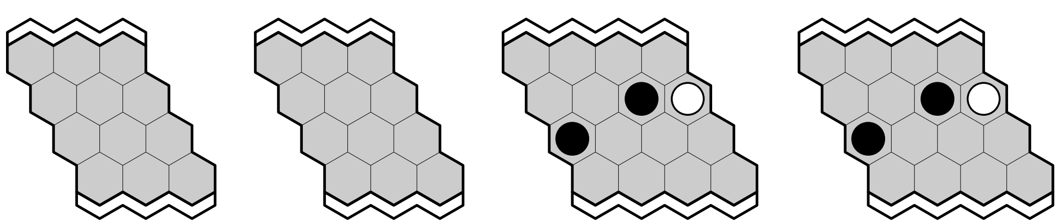
- When did Piet Hein invent Hex?
- Did Gardner believe Nash's invention was independent of Hein's?
- For Hex, give the smallest board with no known particular winning opening move.
- For Rex, give the smallest board with no known particular losing opening move.
- Give $p_1(n)$ for n up to 5.
- Give a strategy-stealing proof of L&S's result.

8 Puzzles

- Hex/Rex, Black/White to play: Best move? Who wins?



- 3×4 CylHex: Black(Around)/White(UpDown) to play: Best move? Who wins?
- CylHex: Black/White to play. Best move? Who wins?



Open

- Who wins c×h Cylindrical Hex, for odd $c \geq 5$?
- Prove/disprove: for n odd, $\lim_{n \to \infty} p_1(n) = .5$.

Truncated Rex (TRex): game ends if only one empty cell (so, draw possible).

State $S = (P, X)$: board position P and player-to-move X.

For a state $S = (P, X)$, define $S' = (P', X)$, where P' is obtained from P by either adding a stone of either color, or removing a stone of either color.

Lemma (Toft):
For TRex, if player Z wins $S = (P, X)$, then Z not-loses S'.

Proof (sketch). Consider a winning strategy for S, and modify it for S': play as in the corresponding position in S; if this is ever not possible (because the move is already occupied), then play anywhere, and argue by induction on the number of empty cells.

Z wins, so the last move in every line of play is by $\overline{Z}$. This is TRex, so there is an empty cell after each last move. Also, just before $\overline{Z}$ makes their final move, every empty cell must be a losing move. So there are at least two such cells in every case, so changing the color of only one of these empty cells to Z's color — which is what happens when we move this strategy to S' — will not give Z a losing position. So Z can at least draw in S'. □

Theorem (Toft):
In TRex, for $S = (P, Y)$ with P player-symmetric, each player non-loses.

Proof. strategy stealing

- Assume Z wins $S = (P, Y)$. — Z is Y or $\overline{Y}$
- Z wins $(P^+, \overline{Y})$ — for a P^+ obtained by adding a Y-stone to P
- Z not-loses $(P, \overline{Y})$ — use the lemma
- $\overline{Z}$ not-loses $(P, Y) = S$ — player-symmetry
- contradiction
- Z not-wins S — original assumption is false
- $\overline{Z}$ not-wins S — Z is arbitrary
- in TRex, each player not-loses. □

Corollary. Winder, Lagarias/Sleator results — empty board player-symmetric

Hayward/Toft/Henderson, How to Play Reverse Hex, DM 312-1 (6 Jan 2012) 148-156
Huneke/Hayward/Toft, A winning strategy for 3×n cylindrical Hex, DM 331 (2014) 93-97

`http://webdocs.cs.ualberta.ca/~hayward/papers`